

## SO(2)-EQUIVARIANT VECTOR FIELDS ON 3-MANIFOLDS: MODULI OF STABILITY AND GENERICITY

GENESIO LIMA DOS REIS AND GEOVAN TAVARES DOS SANTOS

**ABSTRACT.** An open and dense class of vector fields on 3-dimensional compact manifolds equivariant under the action of  $SO(2)$  is defined. Each such vector field has finite moduli of stability. We also exhibit an open and dense subset of the  $SO(2)$ -equivariant gradient vector fields which are structurally stable.

**Introduction and statement of results.** Let  $G = SO(2)$  be the Lie group of rotations acting smoothly and effectively on a smooth three dimensional closed (compact, connected, boundaryless) manifold  $M$ . Let  $\chi^r(G, M)$  denote the space of all  $G$ -equivariant  $C^\infty$  vector field endowed with the  $C^r$  topology. In this paper we define a class of vector fields in  $\chi^r(G, M)$  named  $G$ -Morse-Smale, which is open and dense and prove that each vector field in this class has finite moduli of stability. We also make a similar study for the corresponding class in the  $C^\infty$   $G$ -equivariant gradient vector fields; in this setting we actually have structural stability.

Three main notions play a role in our work.

(a) Finite moduli of stability [P2]. It means that we can parametrize a neighborhood of the vector field by a finite number of  $n$ -parameter families of vector fields.

(b) Normal hyperbolicity.

(c) The third ingredient is  $G$ -transversality of stable and unstable manifolds [F2, B1]. In our case this concept coincides with that of stratumwise transversality. The compactness of  $G$  implies that  $M$  is finitely stratified by  $G$ -invariant submanifolds with each stratum invariant by  $G$ -equivariant vector fields. Stratumwise transversality is the usual transversality inside each of these strata.

Now we state our results in a more precise way. Definitions may be found on §1.

A  $G$ -equivariant vector field on  $M$  is said to be  $G$ -Morse-Smale if

(i) its nonwandering set consists of a finite number of normally hyperbolic critical elements;

(ii) the stable and unstable manifolds of critical elements are  $G$ -transversal.

As it was said before, in the following theorems,  $G$  is  $SO(2)$  and  $\dim(M) = 3$ .

**THEOREM A.** *The set of  $G$ -Morse-Smale vector fields is open in  $\chi^r(G, M)$ .*

**THEOREM B.** *The set of  $G$ -Morse-Smale vector fields is dense in  $\chi^r(G, M)$  if  $r = 1$ . If  $r \geq 2$  and  $M/G$  is orientable, the projective plane, the Klein bottle, or*

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Received by the editors March 31, 1986 and, in revised form, May 5, 1987.

1980 *Mathematics Subject Classification.* Primary 58F10.

*Key words and phrases.* Compact Lie group, equivariant vector field, normal hyperbolicity, stratumwise transversality, moduli of stability.

Partially supported by CNPq (Brasil).

the torus with a cross-cap, then the set of  $G$ -Morse-Smale vector fields is dense in  $\chi^r(G, M)$ .

**THEOREM C.** *If a  $G$ -Morse-Smale vector field has  $l$  saddle connections between dimension one critical elements and  $k$  critical elements of torus type, then its modulus of stability is  $l + k$ .*

A  $C^\infty$  function  $f: M \rightarrow \mathbb{R}$  which is constant along the orbits of the group  $G$  induces, via a bi-invariant metric on  $M$ , a  $G$ -equivariant  $C^\infty$  gradient vector field on  $M$ . The set of all these vector fields is denoted  $\text{Grad}(G, M)$ .

We introduce now a class of vector fields in  $\text{Grad}(G, M)$  called  $G$ -equivariant Morse-Smale gradient vector fields and denoted  $G\text{MSG}$  defined by the following properties:

- (a) The connected components of the zero set of  $X \in G\text{MSG}$  are nondegenerate critical manifolds (in the sense of Bott) [Bo];
- (b) Stable and unstable manifolds of nondegenerate critical manifolds are stratumwise transverse.

**THEOREM D.** (1)  $G\text{MSG}$  is open and dense in  $\text{Grad}(G, M)$ .  
 (2) Each  $X \in G\text{MSG}$  is structurally stable in  $\text{Grad}(G, M)$ .

The contents of this paper are as follows. §1 contains the basic material. In §2 we prove Theorem A essentially following the pattern of the proof for the openness of Morse-Smale vector fields [P1]. Theorem B is proven in §3; known results on the density on two-dimensional manifolds are applied to the projection of the vector field, via quotient map  $M \rightarrow M/G$  and then it is lifted back to  $M$  using a Theorem of Schwartz [Sc]. For the proof of Theorems A and B when  $M$  is two-dimensional and  $G$  is any compact Lie group see [R]. As to Theorem C, linearization on neighborhood of critical elements are assumed (see [H, St, T]). A special case (equal eigenvalues) of a theorem in [H] is applied. The set of  $G$ -equivariant vector fields satisfying these conditions is open and dense in  $\chi^r(G, M)$ . The proof of Theorem C is given in §5, by using the constructions made in §4 which by its turn were inspired by [P2]. Theorem D is proven in §6, and in §7 we give examples of  $G$ -equivariant vector fields with moduli of stability greater than one on any 3-manifold where  $G$  acts.

In this paper we study only the case when  $G = \text{SO}(2)$  and  $M$  is a three-dimensional manifold. Some results for  $G = \mathbb{Z}_2$  are implicit in [Pa]. The case  $\dim(M) = 2$  is studied in [R] for flows, and in [MRM] for diffeomorphisms.

We finish this introduction by stating two problems.

**Problem 1.** To study bifurcation for  $\text{SO}(2)$ -equivariant vector fields on the three dimensional setting. For the two-dimensional case  $G$  is any group, see [MP].

**Problem 2.** Classify the  $\text{SO}(2)$ -Morse-Smale vector fields on three manifolds. See [Pe2] for the classification of Morse-Smale vector fields on two-manifolds.

This paper was done when both authors were visiting the University of California at Berkeley. We take this opportunity to thank the Mathematics Department at that university for the pleasant environment they proportionated to us. We would also like to thank Jacob Palis for his cooperation and stimulating ideas.

**1. Basic definitions and facts.** Here we recall some basic definitions and results. See Bredon [Br], Smale [Sm], and Field [F2] for details.

Let  $G$  be a compact Lie group acting on smooth manifolds  $M$  and  $N$ . A map  $f: M \rightarrow N$  is said to be  $G$ -equivariant if it commutes with the  $G$ -actions, that is,  $f(gx) = g(fx)$  for all  $g \in G$  and  $x \in M$ . The action of  $G$  on  $M$  induces a linear action on the tangent bundle  $TM$  by  $gv = Dg(x)v$ , for  $v \in T_x M$ , where  $Dg(x): T_x M \rightarrow T_{gx} M$  is the differential of the diffeomorphism  $g: M \rightarrow M$  given by  $g \in G$ . An equivariant vector field on  $M$  is an equivariant section  $X: M \rightarrow TM$ . Thus  $X(gx) = Dg(x)X(x)$ . The flow  $X_t$  of  $X$  is equivariant: for each  $t \in \mathbb{R}$ ,  $X_t(gx) = gX_t(x)$  for all  $g \in G$  and  $x \in M$ . So, we can define an action of  $G \times \mathbb{R}$  on  $M$  by  $(g, t)x = X_t(gx)$ . Similarly, if  $f$  is a  $G$ -equivariant diffeomorphism on  $M$ , we have an action of  $G \times \mathbb{Z}$  on  $M$  by  $(g, n)x = f^n(gx)$ . We denote by  $\chi^r(G, M)$  and  $\text{Diff}^r(G, M)$  the spaces of all  $C^\infty G$ -equivariant vector fields and diffeomorphisms, respectively, with the  $C^r$  topology, when  $M$  is compact.

A critical element for  $f \in \text{Diff}^r(G, M)$  is a  $G \times \mathbb{Z}$ -orbit which is compact. Similarly, for  $X \in \chi^r(G, M)$  a critical element is a  $G \times \mathbb{R}$ -orbit  $V$  which is compact. So  $V = (G \times \mathbb{R})(x)$  for some  $x \in M$ . There are three possibilities for  $V$  (the first two may not be exclusive): (a)  $V = R(x)$  = trajectory of  $X$ . (b)  $V = G(x)$ . (c)  $V \neq R(x), G(x)$ ; in this case  $G(x)$  is a global Poincaré section for the restriction of  $X$  to  $V$ . We observe that, in this case, all trajectories of  $X$  starting at  $G(x)$  spend the same time  $\tau$  to meet  $G(x)$  again. We call  $\tau$  the period of  $V$ .

A critical element  $V$  of  $X$  is said to be normally hyperbolic if: (a) there is a continuous splitting  $T_V M = TV \oplus N^u \oplus N^s$  of  $TM$  restricted to  $V$ , preserved by the flow  $X_t$ . That is,  $DX_t \cdot TV = TV$ ,  $DX_t \cdot N^u = N^u$ ,  $DX_t \cdot N^s = N^s$ , for every  $t \in \mathbb{R}$ ; (b) there are numbers  $a < 0 < b$  and  $T > 0$  such that for any  $v \in TV$  and  $t > T$ ,  $e^{at}|v| < |DX_t \cdot v| < e^{bt}|v|$ ; for any  $v \in N^u$  and  $t > T$ ,  $|DX_{-t} \cdot v| \leq e^{bt}|v|$ ; for any  $v \in N^s$  and  $t > T$ ,  $|DX_t \cdot v| \leq e^{at}|v|$ . Here  $|\cdot|$  is a norm induced by a Riemannian metric, which may be taken to be  $G$ -invariant.

$V \subset M$  is called a nondegenerate critical submanifold for a nonconstant  $C^\infty$  function  $f: M \rightarrow \mathbb{R}$  if  $Df(p) = 0$ ,  $p \in V$  and in addition,  $\{v \in T_p M \mid D^2 f(p)(v, w) = 0 \text{ for all } w \in T_p V\} = T_p V$ . In particular  $V$  is a normally hyperbolic critical element for the vector field  $\text{grad}(f)$ .

The following is referred to as the invariant manifold theorem [HPS, F2]: for a normally hyperbolic critical element  $V$  of  $X \in \chi^r(G, M)$  there exist  $C^r G$ -invariant locally  $X$ -invariant submanifolds of  $M$ ,  $W_{\text{loc}}^u(V)$  and  $W_{\text{loc}}^s(V)$ , tangent at  $V$  to  $TV \oplus N^u$  and  $TV \oplus N^s$ , respectively.  $W_{\text{loc}}^u(V)$  (resp.,  $W_{\text{loc}}^s(V)$ ) is characterized as the set of points in a neighborhood  $U$  of  $V$  whose negative (resp. positive) trajectories remain in a neighborhood of  $V$  contained in  $U$ . Moreover,  $W_{\text{loc}}^u(V)$  has the structure of  $C^r$  equivariant locally trivial fibrations over  $V$ . The fiber  $W_{\text{loc}}^u(p)$  at  $p \in V$  is  $C^r$ , its dimension is the dimension of  $N_p^u$ , and it is tangent to  $N_p^u$ .

Also, for a normally hyperbolic critical element  $V$ , there exist an open neighborhood  $Q$  of  $X$  in  $\chi^r(G, M)$  a  $G$ -invariant neighborhood  $U$  of  $V$ , and continuous maps  $F, F^u, F^s: Q \rightarrow \text{Diff}^r(G, M)$  such that:

- (a)  $F(X) = F^u(X) = F^s(X) = \text{identity map of } M$ ;
- (b) for  $X' \in Q$ ,  $F(X')(V) = V'$  is the unique normally hyperbolic critical element for  $X'$  in  $U$ ;

(c)  $F^u(X')(W_{\text{loc}}^{uu}(p)) = W_{\text{loc}}^{uu}(F(X')(p))$ ,  $p \in V$ ,  $X' \in Q$ .

Similarly for  $W_{\text{loc}}^u$ ,  $W_{\text{loc}}^{ss}$ , and  $W_{\text{loc}}^s$ . As usual, the global stable and unstable manifolds  $W^s(V)$  and  $W^u(V)$  are obtained by iterating the local ones by the flow  $X_t$ . For diffeomorphisms, the theory is similar.

The normally hyperbolic critical element  $V$  is said to be a source if  $\dim(W^u(V)) = \dim(M)$ ; a sink if  $\dim(W^s(V)) = \dim(M)$ ; of saddle type if  $0 < \dim(W^s(V)) < \dim(M)$ .

Before defining stratumwise transversality, we recall some facts from compact transformation groups. We say  $x, y \in M$  or their orbits  $G(x), G(y)$  are of the same orbit type if their isotropy groups  $G_x, G_y$  are conjugate:  $G_x = gG_yg^{-1}$ , for some  $g \in G$ , or equivalently, if  $G(x)$  and  $G(y)$  are equivariantly diffeomorphic. Type  $(G(x)) = (H)$  means that  $G_x$  is conjugated to  $H$ . If  $W$  is a submanifold of  $M$ , we denote by  $W_{(H)}$  the submanifold of  $W$  whose points have type  $(H)$ . It is a fact that orbit type partitions  $M$  into a union of submanifolds  $M_{(H)}$ , the union being finite when  $M$  is compact. There is a (principal) stratum  $M_{(H)}$  which is open and dense in  $M$ , consisting of  $G$ -orbits of the highest dimension, called principal orbits.

Now let  $W^s$  and  $W^u$  be stable and unstable manifolds of critical elements of an equivariant vector field or diffeomorphism. We say that  $W^s$  meets  $W^u$  stratumwise transverse at  $x \in M$ , if  $W_{(H)}^s$  is transversal to  $W_{(H)}^u$  in  $M_{(H)}$ , where  $H = G_x$ . We use the notation  $W^s \pitchfork_G W^u$  to indicate stratumwise transversality.

A homeomorphism  $h: M \rightarrow N$  is said to be a conjugacy between two vector fields  $X$  and  $Y$  if  $hX_t = Y_th$ , for all  $t \in \mathbb{R}$ . If we weaken the latter condition by just requiring that  $h$  sends trajectories of  $X$  into trajectories of  $Y$  preserving their orientation, then  $h$  is said to be a topological equivalence. The vector field  $X$  is structurally stable if each  $Y$  near  $X$  (in the  $C^r$  topology) is topologically equivalent to  $X$ . The vector field  $X$  has finite modulus of stability if the topological equivalence classes of vector fields near  $X$  can be expressed by a finite number of  $n$ -parameter families of vector fields.

We finish this section by looking at the case of the group  $G = \text{SO}(2)$  acting effectively on a 3-dimensional manifold  $M$ . For the classification of such actions see [OR]. See also §7, for examples. Let  $G(x)$  be an orbit of dimension 1, i.e.,  $G(x)$  is diffeomorphic to the circle  $S^1: G(x) \sim S^1$ . By the Slice Theorem [Br], there is a closed 2-disc  $S_x$  (the slice at  $x$ ) normal to  $G(x)$  at  $x$ , on which the isotropy group  $G_x$ , a finite cyclic group, acts linearly and effectively.  $T = G(S_x)$  is a tubular neighborhood of  $G(x)$ .

(a) If  $G_x$  is the identity then  $G(x)$  and all the orbits in  $T$  are principal. The action on  $T$  is equivalent to the principal action of  $\text{SO}(2)$  on the solid torus  $D^2 \times S^1: z(z_1, z_2) = (z_1, zz_2)$ .

(b) If  $G_x = Z_p$ , with  $p \geq 2$ , then  $G(x)$  is an exceptional orbit and all the other orbits in  $T$  are principal (it has the dimension of the principal orbits but it has different type). The action on  $T$  is equivalent to the standard linear action of  $\text{SO}(2)$  on  $D^2 \times S^1: z(z_1, z_2) = (z^q z_1, z^p z_2)$ ,  $0 < q < p$  and  $q, p$  relatively prime integers.

(c)  $G_x = Z_2$  acts on  $S_x$  as a reflection around the segment  $\text{Fix}(G_x, S_x)$  (the fixed point set for the  $G_x$  action on  $S_x$ ). All the  $G$ -orbits through points of  $\text{Fix}(G_x, S_x)$  are called special exceptional and denoted SE. The remaining orbits in  $T$  are principal.  $T$  is topologically the product of the Moebius band and an interval. The

action of  $SO(2)$  is trivial on the second factor and standard on the Moebius band (the orbits are circles parallel to the center line). The center line of each Moebius band is an SE orbit.

Now if  $x$  is a fixed point,  $G(x) = \{x\}$ , then a slice  $S_x$  is a 3-dimensional, on which  $G_x = G$  acts as the action on  $D^2 \times I: z(z_1, t) = (zz_1, t)$ .  $\text{Fix}(G, S_x)$  is a 1-manifold. So we have the stratification  $M = P \cup E \cup SE \cup \text{Fix}(G, M)$ , where  $P$  (principal stratum) has dimension 3,  $SE$  (special exceptional) has dimension 2, and both  $E$  (exceptional) and  $\text{Fix}(G, M)$  have dimension 1.

**2. Openness.** The following lemma contains elementary facts which are immediate consequences of the local  $G$ -orbit ( $G = SO(2)$ ,  $\dim M = 3$ ) structure and the  $G$  and  $X$ -invariance of  $W^s$  and  $W^u$ .

**LEMMA (2.1).** *Let  $V_1, V_2$  be normally hyperbolic critical elements of saddle type (i.e.  $0 < \dim W^s(v_i) < 3$ ) of  $X \in \chi^r(G, M)$ . Let  $W$  denote either  $W^s$  or  $W^u$ .*

- (a) *If  $\dim W(V_1) = 1$ , then  $W(V_1) \subset \text{Fix}(G, M)$ .*
- (b) *If  $\dim W(V_1) = 2$ , then  $(W(V_1) - V_1) \cap \text{Fix}(G, M) = \emptyset$ .*
- (c) *If  $W^s(V_1) \cap W^u(V_2) \neq \emptyset$ , then  $\dim W^s(V_1) = \dim W^u(V_2)$ .*
- (d) *Each connected component of  $W(V_1) - V_1$  is a  $(G \times R)$ -orbit.*
- (e) *The connected components of  $W^s(V_1) - V_1$  and  $W^u(V_2) - V_2$  either coincide or are disjoint.*
- (f) *If  $W^s(V_1) \nsubseteq_G W^u(V_2) \neq \emptyset$ , then either  $W^s(V_1), W^u(V_2) \subset \text{Fix}(G, M)$  or  $W^s(V_1), W^u(V_2) \subset SE$ . In each case,  $W^u(V_1) - V_1$  and  $W^s(V_2) - V_2$  are contained in the principal stratum  $P$ .*

**PROOF.** It is enough to examine the  $G$ -structure to have a picture of the stable and unstable manifolds of a critical element of saddle type. For example, for (d): if  $\dim W(V_1) = 1$ , then a component of  $W(V_1) - V_1$  is a trajectory of  $X$  and thus a  $G \times R$ -orbit, since  $W(V_1) \subset \text{Fix}(G, M)$ . If  $\dim W(V_1) = 2$ , choose any  $G$ -orbit  $G(x)$  (necessarily, a circle) in a component of  $W(V_1) - V_1$ . Then that component is the saturation of  $G(x)$  by the flow of  $X$ , and so is a  $G \times R$ -orbit.

**LEMMA (2.2).** *Let  $V$  be a normally hyperbolic critical element of  $X \in \chi^r(G, M)$ . Let  $F, F^s: Q \rightarrow \text{Diff}^r(G, M)$  be the continuous maps given by the invariant manifold theorem, where  $Q$  is a neighborhood of  $X$ . For  $X' \in Q$ , let  $V' = F(X)(V)$  be the corresponding critical element of  $X'$ . If  $M_{(H)}$  is any  $G$ -orbit stratum, then  $W^s(V) \cap M_{(H)} \neq \emptyset$  if and only if  $W^s(V') \cap M_{(H)} \neq \emptyset$ .*

**PROOF.** Set  $f = F^s(X)$ . Then  $f: M \rightarrow M$  is a  $G$ -equivariant map such that  $f(W^s(V)) = W^s(V')$ . Since  $G_{f(x)} = G_x$ ,  $f$  preserves  $G$ -orbit type.

**LEMMA (2.3).** *Let  $V_1, V_2$  be normally hyperbolic critical elements of saddle type of  $X \in \chi^r(G, M)$ . The property  $W^s(V_1) \cap \nsubseteq_G W^u(V_2) \neq \emptyset$  is persistent under small perturbations of  $X$ .*

**PROOF.** It follows from Lemma (2.1)(f) and Lemma (2.2).

Now let  $X$  be a  $G$ -Morse-Smale vector field and let  $V_1, V_2$  be critical elements of saddle type of  $X$  such that  $W^s(V_1) \nsubseteq_G W^u(V_2) \neq \emptyset$ . By Lemma (2.1), it follows that if  $V_0$  is a critical element of  $X$  such that  $W^s(V_0) \cap W^u(V_1) \neq \emptyset$  then  $V_0$  is a sink. Similarly, if  $V_3$  is such that  $W^s(V_2) \cap W^u(V_3) \neq \emptyset$ , then  $V_3$  is a source. These facts may be rephrased in the following way.

Define  $V_1 < V_2$  if and only if  $W^s(V_1) \cap W^u(V_2) \neq \emptyset$ . If  $V_0 < V_1 < V_2 < V_3$ , then  $V_0$  is a sink and  $V_3$  is a source. As a consequence,  $X$  has no cycle (i.e. we cannot have  $V_1 < V_2 < \dots < V_n < V_1$ ). For the proof of the following lemma we may use “filtrations” as in [Sm].

LEMMA (2.4). *A  $G$ -Morse-Smale vector field is  $\Omega$ -stable.*

Let  $X$  be a  $G$ -Morse-Smale vector field and let  $V_1, \dots, V_n$  be its critical elements. Let  $Q$  be a neighborhood of  $X$  such that if  $X' \in Q$  then to each  $V_i$  corresponds a unique critical element  $V'_i$  of  $X'$ , which is normally hyperbolic (invariant manifold theorem). By Lemma (2.4), the nonwandering set of  $X'$  is  $\{V'_1, \dots, V'_n\}$ . This, with Lemma (2.3), proves Theorem A.

**3. Density.** Let  $\pi: M \rightarrow M/G$  be the quotient map. We recall that  $M$  is stratified by  $G$ -orbit strata:  $M = P \cup E \cup \text{SE} \cup \text{Fix}(G, M)$ . The principal stratum  $P$  is open and dense in  $M$ . The restriction of  $\pi$  to each stratum is a differentiable map onto its image, which is a differentiable manifold. So  $M/G$  is a compact topological two-manifold, made up of differentiable manifolds, with boundary  $\partial(M/G) = \pi(\text{SE} \cup \text{Fix}(G, M))$ .

Given  $X \in \chi^r(G, M)$ . We want to approximate  $X$  by a  $G$ -Morse-Smale vector field. By applying the  $G$ -equivariant version of the Kupka-Smale Theorem [F2], we approximate  $X$  by  $Y \in \chi^r(G, M)$  whose critical elements are normally hyperbolic. Denote by  $\pi(Y)$  the vector field on  $M/G$  induced by  $Y$  via  $\pi$ . Note that  $\partial(M/G)$  is invariant by  $\pi(Y)$ .

Let  $V$  be a critical element of  $Y$ . If  $V \subset \text{Fix}(G, M)$ , then  $V$  consists of (one or more) singularities or is a closed trajectory if and only if  $\pi(V)$  consists of (one or more) singularities or is a closed trajectory. If  $V \not\subset \text{Fix}(G, M)$  and  $V \sim S^1$  then  $\pi(V)$  is a singularity of  $\pi(Y)$  irrespective of whether  $V$  consists of singularities or is a closed trajectory of  $Y$ .

Now let  $\gamma = \{Y_t(x)\}$  be a  $\omega$ -recurrent trajectory of  $Y$  through a point  $x \in M$ ,  $\gamma \not\subset G(x)$ . If  $\gamma$  intersects  $G(x)$  again, that is,  $Y_t(x) \in G(x)$  for some  $t \neq 0$ , then  $V = (G \times R)(x)$  is compact (a torus) so it is a critical element for  $Y$ , and  $\pi(\gamma) = \pi(V)$  is a closed trajectory for  $\pi(Y)$ . Conversely, a closed trajectory for a vector field on  $M/G$  lifts to a  $G \times R$ -invariant torus  $V$  on  $M$ , consisting of either closed or dense trajectories on  $V$ . Now, if  $\gamma$  is a  $\omega$ -recurrent trajectory through  $x$  but does not intersect  $G(x)$  again then  $\pi(\gamma)$  is a nontrivial  $\omega$ -recurrent trajectory on  $M/G$ .

Let  $V_1, V_2$  be saddles of  $Y$ , such that  $W^s(V_1) \cap W^u(V_2) \neq \emptyset$ . By Lemma (2.1),  $\dim W^s(V_1) = \dim W^u(V_2)$ . If  $W^s(V_1) \cap W^u(V_2)$  is contained in either  $\text{Fix}(G, M)$  or  $\text{SE}$  then  $W^s(V_1) \not\perp_G W^u(V_2)$ . So, if  $W^s(V_1)$  is not  $G$ -transverse to  $W^u(V_2)$  then their intersection is contained in  $P$ .

Consider now the vector field  $\pi(Y)$  on  $M/G$ . We need to eliminate nontrivial recurrences (see [PM]), which are contained in  $\pi(P)$ . Since the perturbations needed are made away from the boundary, we may apply Peixoto's [Pe1], Markley's [M], and Gutierrez's [G] theorems when  $M/G$  is orientable, the projective plane, the Klein bottle and the torus with a cross-cap, in the  $C^r$  topology. Otherwise, we apply the Closing Lemma [Pu] in the  $C^1$  topology. In this way we get a vector field  $Z^*$  on  $M/G$  which coincides with  $\pi(X_1)$  on  $\partial(M/G)$ . Now, by using a lifting

theorem due to G. Schwartz [Sc], we lift  $Z^*$  to get a  $G$ -Morse-Smale vector field  $Z$  on  $M$ , near  $X$ , concluding the proof of Theorem B.

#### 4. Local constructions.

(I) *Foliations.* By a  $G$  and  $X$  invariant foliation we mean a (continuous) foliation with fibers of class  $C^r$  such that each  $g \in G$  and  $X_t$ ,  $t \in R$ , sends fiber into fiber. Let  $V$  be a critical element of  $X$ . If  $\dim W^s(V) = 1$ , take a  $G$ -invariant 2-disk  $D$  transverse to  $W^s(V)$ . Apply the  $\lambda$ -lemma [P1] to get a  $G$  and  $X$  invariant unstable foliation  $\phi^u$  whose fibers are  $X_t(D)$ ,  $t \in R$ , and  $W^u(V)$ . If  $\dim W^s(V) = 2$ , take a segment  $S$  transverse to  $W^s(V)$  at  $X$  and invariant under  $G_x$ . Now the fibers of  $\phi^u$  are  $gX_t(S)$ ,  $g \in G$ ,  $t \in R$ , and  $W^{uu}(p)$ ,  $p \in V$ . Similarly, we define a stable foliation  $\phi^s$  transverse to  $W^u(V)$ .

(II) *Local moduli of stability.* Here "local" refers to a neighborhood of a critical element. Let  $V$  denote a normally hyperbolic critical element of  $X \in \chi^r(G, M)$ . To  $V$  we associate an invariant  $K(V, X)$  described below.

If  $V = \{x\}$  is an isolated singularity for  $X$ , then  $X$  is structurally stable at  $x$  and we put  $K(V, X) = s$ .

If  $V \sim S^1$  then it is either a closed trajectory for  $x$ , in which case we define  $K(V, X) = cs$ , or  $X|V = 0$  and we set  $K(V, X) = u$ . In the latter case, there are two (local) topological equivalence classes of vector fields near  $X$ :  $\{X_1; X_1|V_1 = 0\}$  and  $\{X_2; V_2 \text{ is closed trajectory of } X_2\}$ ,  $V_1, V_2$  being critical elements of  $X_1, X_2$  near  $V$ .

Suppose now that  $V \sim G(x) \times S^1 \sim S^1 \times S^1$ . Let  $f: G(x) \rightarrow G(x)$  be a Poincaré map for  $X|V$ . Let  $\bar{g} \in G$  such that  $f(x) = \bar{g}x$ . Then, for any  $g \in G$  we have  $f(gx) = gf(x) = g(\bar{g}x) = \bar{g}(gx)$  since  $G$  is abelian. So  $f(y) = \bar{g}y$ , for all  $y \in G(x)$ , i.e.  $f$  is a rotation  $\bar{g} \in G$ . It is easy to see that  $\bar{g}$  depends only on  $V$  and  $X$ , and not on the particular  $G$  orbit. We define  $K(V, X) = \bar{g}$ . Observe that the trajectories of  $X|V$  are either all closed or dense in  $V$  according to whether  $\bar{g}$  is a rational or an irrational rotation. In fact, there is a  $T \in R$  such that  $X_T(G(x)) = G(x)$ .  $T$  is called the period of  $V$ .

Let  $V_0$  be a normally hyperbolic critical element for  $X_0 \in \chi^r(G, M)$ . The invariant manifold theorem gives us a neighborhood  $Q$  of  $X_0$  in  $\chi^r(G, M)$  such that for each  $X' \in Q$  we have a normally hyperbolic critical element  $V'$  equivariantly diffeomorphic and close to  $V_0$ .

**PROPOSITION (4.1).**  *$X, X' \in Q$  are topologically equivalent in a neighborhood of  $V_0$  if and only if  $K(V, X) = K(V', X')$ .*

**PROOF.** The proof in one direction is clear. For the other suppose that  $K(V, X) = K(V', X')$ . We want to define a  $G$ -equivariant conjugacy  $h: W^s(V) \rightarrow W^s(V')$  between  $X|W^s(V)$  and  $X'|W^s(V')$  when  $X$  and  $X'$  are conveniently parametrized (the construction for  $W^u$  is similar).

If  $X|V = 0$  (so  $X'|V' = 0$ ), let  $x_1, x_2$  and  $x'_1, x'_2$  be points in distinct components of  $W^s(V) - V$  and  $W^s(V') - V'$  (if there is only one component, one point is enough). We put  $h(x_1) = x'_1$ ,  $h(x_2) = x'_2$  and extend  $h$  to  $W^s(V)$  by conjugacy ( $h = X'_t h X_t$ ) and equivariance ( $h = g^{-1} h g$ ). (See Lemma 2.1(d).) Then we extend  $h$  to a neighborhood of  $V$  by continuity. This conjugacy  $h$  is well defined and continuous.

Now suppose  $V = G(p)$ ,  $V' = G(p')$  are closed trajectories. Let  $\Sigma \subset W^s(V)$ ,  $\Sigma' \subset W^s(V')$  be  $G_p$ -invariant sections transverse to  $X$ ,  $X'$  through  $p$ ,  $p'$ . Let  $\tau$  be the period of  $V$ . For each  $t \in \mathbb{R}$  let  $g_t \in G$  be such that  $X_t(p) = g_t p$ . After  $G$ -equivariant reparametrizations we may assume that  $X'_t(P') = g_t p'$  and  $X_\tau(\Sigma) \subset \Sigma$ ,  $X'_\tau(\Sigma') \subset \Sigma'$ . Now we choose  $x_1, x_2 \in \Sigma$ ,  $x'_1, x'_2 \in \Sigma'$  and define  $h$  as before.

Assume now  $V \sim S^1 \times S^1$  is a sink. Let  $f: \Sigma \rightarrow \Sigma$  and  $f': \Sigma' \rightarrow \Sigma'$  be Poincaré maps, where  $\Sigma, \Sigma'$  are  $G$ -equivariant sections transverse to  $X, X'$ . After  $G$ -equivariant reparametrizations we may assume that  $V$  and  $V'$  have the same period  $T$  and  $X_T(\Sigma) \subset \Sigma$ ,  $X'_T(\Sigma') \subset \Sigma'$ . Let  $G(x_0) = \Sigma \cap V$  and  $G(x'_0) = \Sigma' \cap V'$  with  $f|G(x_0) = \bar{g} \in G$ ,  $f'|G(x'_0) = \bar{g}' \in G$ . Now take  $D \subset \Sigma$ ,  $D' \subset \Sigma'$   $G$ -invariant fundamental domains for  $f, f': D$ , and similarly  $D'$ , can be constructed by taking a  $G$ -invariant annulus  $A$ , containing  $G(x_0)$  and putting  $D = \overline{A - f(A)}$ . Let  $W^{ss}(x_0)$ ,  $W^{ss}(x'_0)$  be the strong stable manifolds for  $f, f'$  at  $x_0, x'_0$ . If  $[p, q] = D \cap W^{ss}(x_0)$  and  $[p', q'] = D' \cap W^{ss}(x'_0)$ , let  $h: [p, q] \rightarrow [p', q']$  be a homeomorphism such that  $h(p) = p'$  and  $h(q) = q'$ . We extend  $h$  to  $D$  by equivariance: given  $x \in D$ , let  $g \in G$  be such that  $gx \in [p, q]$  and define  $h(x) = g^{-1}hg(x)$ . To extend  $h$  to  $\Sigma - G(x_0)$  we do it by conjugacy: given  $y \in \Sigma - G(x_0)$  let  $n \in \mathbb{Z}$  such that  $f^n(y) \in D_1$  and put  $h(y) = (f')^{-n}hf^n(y)$ . Now put  $h(x_0) = x'_0$  and extend it to  $G(x_0)$  by equivariance. Since  $\bar{g} = \bar{g}'$  we have  $f'h = hf$ . Finally we extend  $h$  to a conjugacy between  $X$  and  $X'$  in a neighborhood of  $V$  via the equation  $h = X'_t h X_t$ . The continuity of  $h$  follows from the fact that it sends strong stable manifolds of  $f$  into strong stable manifolds of  $f'$  (which are one-dimensional). This ends the proof of the proposition.

We observe here that as a consequence of Proposition 4.1 if the normally hyperbolic critical element of  $X$  is a torus then its local modulus of stability is one. Otherwise it is zero.

(III) *Saddle connections.* Let  $V_1, V_2$  be critical elements of saddle type for  $X \in \chi^r(G, M)$  such that  $W^s(V_1) \nparallel_G W^u(V_2) \neq \emptyset$ . By Lemma 2.1(f), the connected components of  $W^s(V_1) - V_1$  and  $W^u(V_2) - V_2$  which intersect must coincide; we call this intersection a saddle connection between  $V_1$  and  $V_2$ .

Let  $V_1 = p_1$ ,  $V_2 = p_2$  be isolated singularities of  $X$ , thus,  $\dim W^s(p_1) = \dim W^u(p_2) = 1$ . Let  $\lambda_1(X) = a_1(X) \pm b_1(X)i$  ( $\lambda_2(X) = a_2(X) \pm b_2(X)i$ ) be the expanding (contracting) eigenvalues of  $X$  at  $p_1$  ( $p_2$ ),  $a_1(X) > 0$ ,  $a_2(X) < 0$ . When the expanding (contracting) eigenvalues are real, by  $G$ -equivariance, they are equal. Let  $\delta_j(X) = b_j(X)/a_j(X)$ ,  $j = 1, 2$ .

Let  $z_0 \in W^s(p_1) \cap W^u(p_2)$ . Take  $V$  a slice for the  $G$ -action at  $z_0$ . Thus  $V$  is equivariantly diffeomorphic to a cylinder where  $G$  acts linearly with fixed points (as described in §1). Choose coordinates  $(y_1, y_2, y_3)$  on  $V$ , such that  $W^s(p_1) \cap W^u(p_2) \cap V = \{y_1 = y_2 = 0\}$ ,  $z_0 = (0, 0, 0)$ , and take  $\Sigma = \{y_3 = 0\}$ . Define  $W^{\delta_1(X)}: \Sigma \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  by  $W^{\delta_1(X)}(y_1, y_2) = e^{i(\delta_1(x) \log r + \phi)}$ , where  $(y_1, y_2) = (r \cos \phi, r \sin \phi)$ . (Here we are identifying  $\mathbb{R}^2$  with the complex plane). For  $c \in \mathbb{R}/2\pi\mathbb{Z}$  and  $\delta_1(X) \neq 0$ ,  $(W^{\delta_1(X)})^{-1}(c) = \{e^{-t/\delta_1(X)} e^{i(t+c')}, t \in \mathbb{R}\}$ , where  $c = e^{ic'}$ , is a spiral. For  $\delta_1(X) = 0$ ,  $(W^{\delta_1(X)})^{-1}(c)$  is a straight line.

Let  $U_1$  be a  $G$ -invariant neighborhood of  $p_1$  and take a  $G$ -orbit  $\Sigma_1 \subset W^u(p_1) \cap U_1$  as a fundamental domain for  $X$ . Let  $\pi_1(X): U_1 \rightarrow W^u(p_1)$  be a projection which maps trajectories of  $X$  onto trajectories of  $X|W^u(p_1)$ . If the neighborhood  $V$  of  $z_0$  is small enough, for each  $z \in V - W^s(p_1)$  the trajectory of  $X$  through  $z$

intersects  $\pi_1(X)^{-1}(\Sigma_1)$  at a point  $y$ . Define a map  $S_1(X): V - W^s(p_1) \rightarrow \Sigma_1$  by  $S_1(X)(z) = \pi_1(X)(y)$ .

With the notation above we have

LEMMA (4.1). *Let  $W^{\delta_1(X)}: \Sigma \rightarrow R/2\pi Z$  be the map as before. There are a  $G$ -equivariant continuous projection  $\pi_1(X): U \rightarrow W^u(p_1)$  and a diffeomorphism  $v_1: \Sigma_1 \rightarrow R/2\pi Z$  such that  $v_1 \circ S_1(X)(y_1, y_2) = W^{\delta_1(X)}(y_1, y_2)$ . The map  $S_1(X)$  is also  $G$ -equivariant.*

Except for the  $G$ -equivariance, the proof of this lemma is found in [Str]. It is not difficult to get the maps equivariant.

Observe that for  $s \in \Sigma_1$ ,  $S_1(X)^{-1}(s)$  is a cylinder which intersects  $W^u(p_1)$  along a trajectory of  $X$  and  $\Sigma$  along  $(W^{\delta_1(X)})^{-1}(v_1(s))$ .

PROPOSITION (4.2). *If  $X \in \chi^r(G, M)$  has a saddle connection  $\gamma = W^s(p_1) \bar{\cap} W^u(p_2)$  between the singularities  $p_1$  and  $p_2$ , then  $X$  has zero modulus of stability in a neighborhood of  $\text{cl}(\gamma)$ .*

PROOF. (a)  $\delta_1(X) \neq \delta_2(X)$ . First suppose both  $\delta_1(X), \delta_2(X) \neq 0$ . Let  $Q$  be a neighborhood of  $X$  as in the proof of Lemma (2.4) and such that  $\delta_1(Y) \neq \delta_2(Y)$  if  $Y \in Q$ . Let  $p'_1, p'_2$  be the corresponding saddle points for  $Y$ . As we know,  $W^s(p_1), W^u(p_2), W^s(p'_1), W^u(p'_2)$  are contained in the same one-dimensional  $G$ -stratum. For  $z_0 \in W^s(p_1) \cap W^s(p'_1)$ , take  $V$  a  $G$ -invariant neighborhood of  $z_0$ , a slice as above, with coordinates  $(y_1, y_2, y_3)$ , and  $\Sigma$  the plane  $y_3 = 0$ . As in Lemma (4.2) consider the maps  $W^{\delta_1(X)}, W^{\delta_2(X)}, W^{\delta_1(Y)}, W^{\delta_2(Y)}: \Sigma \rightarrow R/2\pi Z$ , and for  $c \in R/2\pi Z$  the spirals  $\Gamma_j(X) = (W^{\delta_j(X)})^{-1}(c) = \{\phi_j(t); t \in R\}$ ,  $\Gamma_j(Y) = (W^{\delta_j(Y)})^{-1}(c) = \{\psi_j(t); t \in R\}$ ,  $j = 1, 2$ . We start by defining a  $G$ -equivariant homeomorphism  $h: \Sigma \rightarrow \Sigma$  which sends  $\Gamma_1(X)$  onto  $\Gamma_1(Y)$  and  $\Gamma_2(X)$  onto  $\Gamma_2(Y)$ . Let  $a_k(X) = 2\pi k \delta_2(X) / (\delta_2(X) - \delta_1(X))$ ,  $k \in Z$ . For each  $k$  we define  $h$  mapping the interval  $\{\phi_1(t); t \in [a_k(X), a_{k+1}(X)]\}$  onto the interval  $\{\psi_1(t); t \in [a_k(Y), a_{k+1}(Y)]\}$  in the following way: for each  $\phi_1(t)$ , there exists a unique  $g \in G$  such that  $g\phi_1(t) \in \Gamma_2(X)$ ; precisely  $g = e^{i\theta}$ , with  $\theta = (\delta_1(X) - \delta_2(X)t) / \delta_2(X)$ . Define  $h(\phi_1(t))$  as being the point  $\psi_1(s)$ ,  $s \in [a_k(Y), a_{k+1}(Y)]$  such that  $g\psi_1(s) \in \Gamma_2(Y)$ . Now extend  $h$  to  $\Sigma$  by  $G$ -equivariance:  $gh(x) = h(gx)$ .

To extend  $h$  to a neighborhood of  $\text{cl}(\gamma)$  we take  $G$ -invariant Liapunov functions  $f_1, f_2, g_1, g_2$  at  $p_1, p_2, p'_1, p'_2$  for  $X$  and  $Y$ . We may assume that  $f_1(\Sigma) = f_2(\Sigma) = g_1(\Sigma) = g_2(\Sigma) = -1$ . We extend  $h$  by requiring that it sends trajectories of  $X$  onto trajectories of  $Y$  and level surfaces of the Liapunov functions for  $X$  onto the same level surfaces of the corresponding Liapunov functions for  $Y$ . Since  $h$  sends cylinders  $S_1(X)^{-1}(s_1)$  (resp.  $S_2(X)^{-1}(s_2)$ ) onto cylinders  $S_1(Y)^{-1}(s'_1)$  (resp.  $S_2(Y)^{-1}(s'_2)$ ), it extends continuously to  $W^u(p_1)$  and  $W^s(p_2)$ .

If  $\delta_1(X) = 0$  or  $\delta_2(X) = 0$  take  $a_k(X) = 2\pi k$ , and proceed as before.

(b)  $\delta_1(X) = \delta_2(X)$ . In this case there are two different topological equivalence classes in a neighborhood of  $X$ :  $\{Y; \delta_1(Y) = \delta_2(Y)\}$  and  $\{Y; \delta_1(Y) \neq \delta_2(Y)\}$ . Let  $Q$  be a neighborhood of  $X$  as in the proof of Lemma (2.4). Let  $Y \in Q$  such that  $\delta_1(Y) = \delta_2(Y)$ . Then  $\Gamma_1(X) = \Gamma_2(X)$  and  $\Gamma_1(Y) = \Gamma_2(Y)$ . Choose any  $G$ -equivariant homeomorphism  $h: \Sigma \rightarrow \Sigma$  sending  $\Gamma_1(X)$  onto  $\Gamma_1(Y)$  and extend it as in (a).

Now assume  $\delta_1(Y) \neq \delta_2(Y)$ . We will show that  $Y$  is not topologically equivalent to  $X$ . In fact, assume there is an equivalence  $h$  between  $X$  and  $Y$ . Let  $z_0 \in W^s(p_1) \cap W^u(p_2)$  and take  $\Sigma$  as in part (a). In a coordinate system near  $z'_0 = h(z_0)$  consider

the curves  $\Gamma_1(Y), \Gamma_2(Y) \subset \Sigma'$  ( $\Sigma'$  is a plane through  $z'_0$ ). The homeomorphism  $h$  maps the cylinder  $S_1(X)^{-1}(s)$ ,  $s \in \Sigma$ , which intersects  $\Sigma$  along  $\Gamma_1(X)$  onto a cylinder which intersects  $\Sigma'$  along a curve  $\Gamma'$ . Projection along trajectories of  $Y$  gives a bijection  $f: \Gamma' \rightarrow h(\Gamma_1(X))$ . See Figure 1.

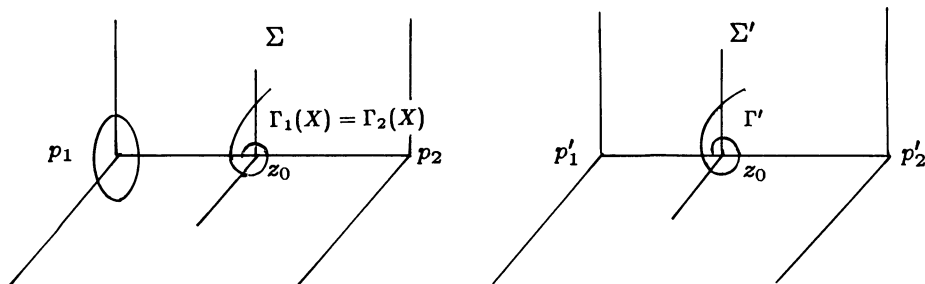


FIGURE 1

We claim that, near  $z'_0$ , the curve  $\Gamma'$  lies between two curves  $(W^{\delta_1(Y)})^{-1}(c_1)$ ,  $(W^{\delta_1(Y)})^{-1}(c_2)$ ,  $c_1, c_2 \in R/2\pi Z$ . Let  $C$  be the cylinder  $\pi_1(Y)^{-1}(h(\Sigma_1))$ . Then  $\Gamma'' = \{X_t(\Gamma'), t \geq 0\} \cap C$  is a curve which tends to  $h(s)$ . Near  $h(s)$ ,  $\Gamma''$  lies between two segments  $L_1, L_2 \subset C$  which intersects  $h(\Sigma_1)$  in points  $s_1, s_2$ . Then  $\Gamma'$  lies between the curves  $S_1(Y)^{-1}(s_1) \cap \Sigma' = (W^{\delta_1(Y)})^{-1}(c_1)$ ,  $S_1(Y)^{-1}(s_2) \cap \Sigma' = (W^{\delta_1(Y)})^{-1}(c_2)$ , proving the claim. Thus the curve  $\Gamma'$  intersects the family  $(W^{\delta_2(Y)})^{-1}(c)$ ,  $c \in R/2\pi Z$  infinitely many times. Take sequences  $x_n \rightarrow z'_0$ ,  $y_n \rightarrow z'_0$ ,  $x_n \in \Gamma' \cap (W^{\delta_2(Y)})^{-1}(c_1)$ ,  $y_n \in \Gamma' \cap (W^{\delta_2(Y)})^{-1}(c_2)$ ,  $c_1 \neq c_2 \pmod{2\pi}$ . Iterating by the flow  $Y_t$  we can get sequences  $Y_{t_n}(x_n)$ ,  $Y_{t_n}(y_n)$  converging to distinct points  $x, y$  in  $h(\Sigma_2)$ . But since  $h^{-1}f(x_n), h^{-1}f(y_n) \in \Gamma_2(X)$  the sequences  $h^{-1}(X_{t_n}(x_n))$ ,  $h^{-1}(X_{t_n}(y_n))$  should converge to the same point in  $\Sigma_2$ , which is absurd.

After what was done above we can define

(i)  $K(V_1, V_2, X) = ss$  if  $V_1$  and  $V_2$  are singularities of  $X$  with a saddle connection between them.

Now suppose  $V_1 \sim V_2 \sim S^1$  we put

(ii)  $K(V_1, V_2, X) = uu$  if  $K(V_1, X) = u$  and  $K(V_2, X) = u$ .

(iii)  $K(V_1, V_2, X) = ucs$  if  $K(V_1, X) = u$  and  $K(V_2, X) = cs$ .

(iv)  $K(V_1, V_2, X) = csu$  if  $K(V_1, X) = cs$  and  $K(V_2, X) = u$ .

(v)  $K(V_1, V_2, X) = \log |\lambda_1| / \log |\mu_2|$  if both,  $V_1$  and  $V_2$ , are closed trajectories of  $X$ , where  $\lambda_1$  is the expanding eigenvalue for a Poincaré map for  $V_1$  and  $\mu_2$  is the attracting eigenvalue for a Poincaré map for  $V_2$ .

**PROPOSITION (4.3).** *Let  $X_0 \in \chi^r(G, M)$  be a vector field with a saddle connection between the saddles  $V_1^0$  and  $V_2^0$  which are diffeomorphic to  $S^1$ . There is a neighborhood  $Q$  of  $X_0 \in \chi^r(G, M)$  such that  $X, X' \in Q$  are topologically equivalent in a neighborhood of  $W^s(V_1^0) \cap W^u(V_2^0)$  if and only if  $K(V_1, V_2, X) = K(V'_1, V'_2, X')$ .*

**PROOF.** Let  $V_1, V_2, V'_1, V'_2$  be critical elements of saddle type for  $X, X'$  such that  $K(V_1, V_2, X) = K(V'_1, V'_2, X')$  and  $W^s(V_1) \cap W^u(V_2) \neq \emptyset$ ,  $W^s(V'_1) \cap W^u(V'_2) \neq \emptyset$ . Let  $\Gamma_1, \Gamma_2, \Gamma'_1, \Gamma'_2$  be neighborhoods of  $V_1, V_2, V'_1, V'_2$ . By Proposition (4.1),

we may suppose there is a  $G$ -equivariant conjugacy  $h: \Gamma_1 \rightarrow \Gamma'_1$  between  $X, X'$ . We want to extend  $h$  to a neighborhood of  $W^s(V_1) \cap W^u(V_2)$ . First of all, we observe that  $h$  is automatically defined in  $W^s(V_1) \cap (W^u(V_2) - V_2)$ , by Lemma (2.1)(e). Assuming that  $X, X'$  have suitable parametrizations, we extend  $h$  to  $T_2$ , as in (II) using the stable foliations near  $V_2$ . These stable foliations are obtained by iterating negatively the unstable foliations of  $V_1$  by the flows  $X_t, X'_t$ . Now the continuity of  $h$  in points of  $W^s(V_2)$  is not clear and depends strongly on the relation  $K(V_1, V_2, X) = K(V'_1, V'_2, X')$ . Let  $w \in W^s(V_2)$  and let  $w_n \in T_2$  be a sequence such that  $w_n \rightarrow w$ . We need to prove that  $h(w_n)$  converges to a point of  $W^s(V'_2)$ . Among the possibilities for  $K(V_1, V_2, X)$  we show the proof for case (v).

We consider case (v), where

$$K(V_1, V_2, X) = \log |\lambda_1| / \log |\mu_2| = \log |\lambda'_1| / \log |\mu'_2| = K(V'_1, V'_2, X').$$

Here,  $V_1, V_2, V'_1, V'_2$  are closed trajectories. Let  $f_1: \Sigma_1 \rightarrow \Sigma_1, f_2: \Sigma_2 \rightarrow \Sigma_2, f'_1: \Sigma'_1 \rightarrow \Sigma'_1, f'_2: \Sigma'_2 \rightarrow \Sigma'_2$  be Poincaré maps for  $X$  and  $X'$  at  $p_1 \in V_1, p_2 \in V_2, p'_1 \in V'_1, p'_2 \in V'_2$ , where  $\Sigma_1, \Sigma'_1$  and  $\Sigma_2, \Sigma'_2$  are  $G_{p_1}$  and  $G_{p_2}$  invariant, respectively. Then  $\lambda_1$  is an eigenvalue of  $f_1$  at  $p_1$ , with  $|\lambda_1| > 1, \mu_2$  is an eigenvalue of  $f_2$ , with  $|\mu_2| < 1$ , and similarly for  $\lambda'_1, \mu'_2$ . Without loss of generality we may assume  $\lambda_1, \mu_2, \lambda'_1, \mu'_2 > 0$ . Let  $D_1 \subset W^s(p_1), D'_1 = h(D_1)$  and  $D_2 \subset W^u(p_2), D'_2 = h(D_2)$  be fundamental domains for  $f_1, f'_1$  and  $f_2, f'_2$ . After  $G$ -invariant reparametrizations we are assuming that  $V_1, V_2, V'_1, V'_2$  have the same period  $\tau, X_\tau(\Sigma_i) \subset \Sigma_i, X'_\tau(\Sigma'_i) \subset \Sigma'_i, h(\Sigma_i) \subset \Sigma'_i$  ( $i = 1, 2$ ) and  $X_s(U_1) \subset \Sigma_2, X'_s(U_1) \subset \Sigma'_2$ , for some  $s \in \mathbb{R}$ , where  $U_1 \subset \Sigma_1, U'_1 \subset \Sigma'_1$  are neighborhoods of  $D_1, D'_1$ . Let  $x_n \in \Sigma_1, y_n \in U_1, z_n \in \Sigma_2$  such that  $x_n \rightarrow x \in W^u(p_1), y_n \rightarrow y \in D_1, z_n \rightarrow z \in W^u(p_2)$  and  $y_n = f_1^{N_n}(x_n), z_n = x_s(y_n)$  and  $w_n = f_2^{M_n}(z_n)$ , where  $N_n, M_n$  are integers. Let  $a, a_n$  be the coordinates of  $x, x_n$  in the  $W^u(p_1)$  direction and  $d, d_n$  be the coordinates of  $w, w_n$  in  $W^s(p_2)$  direction with  $a, a_n, d, d_n > 0$ , in coordinate systems that linearize  $f_1$  and  $f_2$ . We get  $d_n = a_n \lambda_1^{N_n} k_n \mu_2^{M_n}$ , where  $k_n$  is a sequence that converges for a positive number  $k$ . This expression, with the corresponding for the sequences  $x'_n = h(x_n), y'_n = h(y_n), z'_n = h(z_n), w'_n = h(w_n)$ , gives

$$\frac{\log(d'_n/k'_n a'_n)}{\log(d_n/k_n a_n)} = \frac{N_n \log \lambda'_1 + M_n \log \mu'_2}{N_n \log \lambda_1 + M_n \log \mu_2} = \frac{\log \lambda'_1}{\log \lambda_1},$$

proving that  $d'_n$  converges. For a proof of the converse, see [Pa]. This ends the proof of Proposition (4.2).

As a consequence of the last proposition, the existence of a saddle connection  $\Xi$  for a vector field  $X \in \chi^r(G, M)$  implies that it has modulus of stability one when restricted to a neighborhood of  $\Xi$ .

REMARK. For the linearization of the vector field  $X$  on a neighborhood of  $V$  we apply the following theorems: in the case  $K(V, X) = s$ , Theorem (IV) (with (8.3) in [H]); in the case  $K(V, X) = cs$ , the same theorem applied to a Poincaré map for  $X$ ; in the case  $K(V, X) = u$ , Takens' results in [T].

**5. Moduli of stability.** Let  $X$  be a  $G$ -Morse-Smale vector field. To  $X$  and its critical elements  $V_1, \dots, V_n$  ( $k$  among them being tori) are associated invariants  $K(V_1, X), \dots, K(V_n, X)$ . Let  $\Sigma_i = \{V_{i,1}, \dots, V_{i,k(i)}\} \subset \{V_1, \dots, V_n\}, i = 1, 2, \dots, l$ , be the saddle connections of  $X$ , where  $V_{i,j}, V_{i,j+1}$  are consecutive saddles. If  $K(V_{i,j}, V_{i,j+1}, X) = K(V_{i,j+1}, V_{i,j+2}, X), j = 1, \dots, k(i) - 2$ , we can define the

invariants  $K(\Sigma_i, X) = K(V_{i,j}, V_{i,j+1}, X)$ . Now define  $K(X) = (K(V_1, X), \dots, K(V_n, X), K(\Sigma_1, X), \dots, K(\Sigma_l, X))$ .

Now let  $F: Q \rightarrow \text{Diff}^r(G, M)$  be given by the Invariant Manifold Theorem, where  $Q$  is a neighborhood of  $X$ . For  $X_1, X_2 \in Q$ , their critical elements are  $F(X_1)(V_i)$  and  $F(X_2)(V_i)$ ,  $i = 1, 2, \dots, n$ .

PROOF OF THEOREM C. To prove the theorem it is enough to show that  $X_1$  and  $X_2$  are topologically equivalent if and only if  $K(X_1) = K(X_2)$ .

Since one direction of proof is clear suppose  $K(X_1) = K(X_2)$ . The construction of the topological equivalence between  $X_1$  and  $X_2$  will consist in globalization of the local construction of the last section. All the  $G$ -equivariant reparametrizations needed to do that are assumed. We will get a conjugacy  $h: M \rightarrow M$  between reparametrized vector fields  $X_1$  and  $X_2$ .

Let  $V_0^1, V_0^2$  be sinks and let  $V_1^1, V_1^2$  be critical elements of saddle type of  $X_1, X_2$  such that  $K(V_0^1, X_1) = K(V_0^2, X_2)$ ,  $K(V_1^1, X_1) = K(V_1^2, X_2)$  and  $W^s(V_0^1) \cap W^u(V_1^1) \neq \emptyset$ ,  $W^s(V_0^2) \cap W^u(V_1^2) \neq \emptyset$ . Let  $T_0^1, T_0^2$  be compact  $G$ -invariant neighborhoods of  $V_0^1, V_0^2$  and let  $C^1 \subset \partial T_0^1$ ,  $C^2 \subset \partial T_0^2$  be fundamental domains for the  $G$ -action on  $\partial T_0^1, \partial T_0^2$  ( $C^1, C^2$  are arcs containing one and only one point of each  $G$ -orbit). We want to extend a  $G$ -equivariant conjugacy (constructed as in §4)  $h: W^s(V_1^1) \rightarrow W^s(V_1^2)$  between  $X_1$  and  $X_2$  to a neighborhood of  $V_1^1$ . Let  $\phi^u(X_1), \phi^u(X_2)$  be unstable foliations (as constructed in §4). Let  $I \subset C^1$  be a small segment containing  $W^u(V_1^1) \cap C^1$ . Figure 2 represents schematically the situation we have in mind. Given  $x \in I$ , let  $F^1 \in \phi^u(X_1)$  be the fiber containing  $x$  and let  $y = F^1 \cap W^s(V_1^1)$ . Let  $F^2 \in \phi^u(X_2)$  be the fiber through  $h(y)$ . We define  $h(x) = F^2 \cap C^2$  and we extend  $h$  to  $G(I)$  by  $G$ -equivariance. Finally, if  $z$  is a point in a neighborhood of  $V_1^1$  such that  $(X_1)_t(z) \in G(I)$  for some  $t \in \mathbb{R}$  we put  $h(z) = (X_2)_{-t}h(X_1)_t(z)$ .

Doing this construction for each saddle we get  $h$  defined in finitely many intervals contained in  $C^1$ . Since  $h$  can be constructed to be near the identity it can be extended to  $C^1$  and then to  $\partial T_0^1$  to be  $G$ -equivariant.

Now let  $V_2^1$  be a critical element of saddle type such that  $W^s(V_1^1) \Psi_G W^u(V_2^1) \neq \emptyset$ . Then necessarily  $W^s(V_2^1)$  comes from a source  $V_3^1$ .

Proceeding as in §4 and as above, we get  $h$  defined in  $\partial T_3^1$ , boundary of a neighborhood  $T_3^1$  of  $V_3^1$ . In this way, we eventually get  $h$  defined in all  $M$ , except

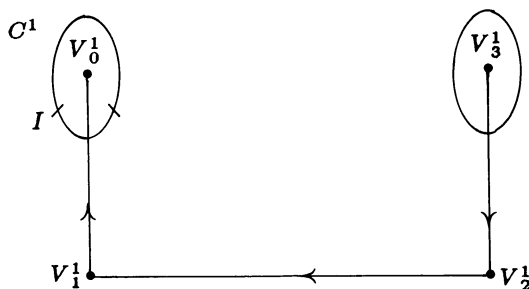


FIGURE 2

in some neighborhoods of sinks and sources. We omit the details for extending  $h$  to such neighborhoods. (See [PM] for such constructions.)

If there is no critical element of saddle type to start with, the construction of  $h$  is easier, and we omit details.

**6.  $G$ -equivariant gradient vector fields.** In this section we prove Theorem D.

(a) *Openness and density.* Let  $GMSG_1$  be the set of all vector fields in  $\text{Grad}(G, M)$  which satisfy property (a) in the definition of  $GMSG$ . It is known that  $GMSG_1$  is open and dense in  $\text{Grad}(G, M)$  [W, p. 149]. Since the vector fields in  $\text{Grad}(G, M)$  has no nontrivial recurrence to prove that  $GMSG$  is dense in  $GMSG_1$  it is enough to show that we can break nonstratumwise transverse saddle connections.

The principal stratum  $P$  is the only one where such phenomenon can occur. If  $\pi: M \rightarrow M/G$  is the projection of  $M$  onto its orbit space and if  $X = \text{grad } f \in GMSG_1$  has saddle connections in  $P$ , then  $\pi(X) = \text{grad } \pi(f)$  has saddle connections in  $\pi(P)$  (here we define  $\pi(f)(x) = f(\pi^{-1}(x))$  and we put in  $M/G$  the natural metric induced by  $\pi$  and a  $G$  invariant metric on  $M$ ). By modifying  $\pi(f)$  in neighborhoods of chosen points on the saddle connections we obtain a function  $\bar{g}: M/G \rightarrow \mathbb{R}$ ,  $\bar{g} = \pi(f)$  outside those neighborhoods, such that  $Y = \text{grad } g$ ,  $g = \bar{g} \circ \pi$ , is close to  $X$  and belongs to  $GMSG$ .

To end the proof we observe that the openness of stratumwise transversality follows from the fact that  $GMSG$  is a subset of the  $G$ -Morse-Smale vector fields.

(b) *Structural stability.* To prove the structural stability for vector fields in  $GMSG$  we need the next proposition.

**PROPOSITION (6.1).** *If  $X = \text{grad } f \in GMSG$  has a saddle connection  $C$  either along  $\text{Fix}(G, M)$  or SE then  $X$  is structurally stable in a neighborhood of  $C$ .*

**PROOF.** It follows from Propositions (4.2) and (4.3).

**7. Examples.** (a) *Basic constructions on  $D^2 \times S^1$ .* Let us represent  $D^2 \times S^1$  as  $D^2 \times [0, 1]$  with  $D^2 \times 0$  identified with  $D^2 \times 1$ . Figures 3(a), (b) give the orbits on  $D^2 \times S^1$  of the action with fixed points  $z(z_1, z_2) = (zz_1, z_2)$  and the principal action  $z(z_1, z_2) = (z_1, zz_2)$ .

Now we will define a vector field on  $D^2 \times S^1$  which will be equivariant under the action with fixed points. First let us take a vector field  $\bar{X}_0$  on  $D^2$  which has

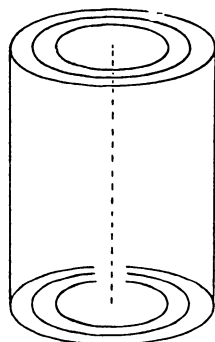


FIGURE 3(a)

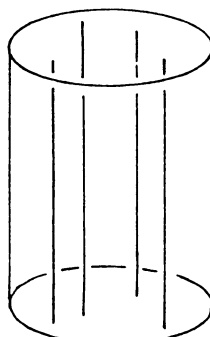


FIGURE 3(b)

one hyperbolic source at zero,  $\bar{X}_0(zz_1) = \bar{X}_0(z_1)$ ,  $z \in G$ ,  $\bar{X}_0 \mid \partial D^2 = 0$  and  $\partial D^2$  is a normally hyperbolic set for  $\bar{X}_0$ . The vector field on  $D^2 \times S^1$  given by  $X_0(z_1, z_2) = (\bar{X}_0(z_1), 0)$  has  $0 \times S^1$  and  $\partial(D^2 \times S^1)$  as normally hyperbolic sets with  $X_0 \mid \partial(D^2 \times S^1) = 0$ . Now we take  $\varepsilon > 0$  small and we define on  $D^2 \times S^1$  the vector field  $E_0(z_1, z_2) = (0, \varepsilon iz_2)$ . All trajectories of  $E_0$  are closed and of the form  $\{z_1\} \times S^1$ ,  $z_1 \in D^2$ . Finally we define  $X = X_0 + E_0$  (see Figure 4) which has the following properties:

(1)  $X$  is equivariant under the action with fixed points since  $X_0$  and  $E_0$  are both equivariant.

(2)  $\{0\} \times S^1$  is a hyperbolic closed orbit (a source) and  $\partial(D^2 \times S^1)$  is a normally hyperbolic set (a sink) [HPS].

(3) The longitudes of  $(D^2 \times S^1)$  are trajectories of  $X$  and then transverse to the  $SO(2)$  orbits.

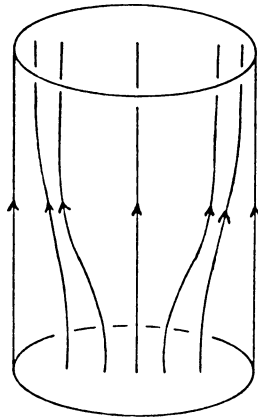


FIGURE 4

We will also define a second vector field equivariant under the action with fixed points. For this we take on the rectangle  $\{(re^{i\theta}, s); 0 \leq r \leq 1, \theta = \beta_0 \text{ or } \theta = -\beta_0, 0 \leq s \leq 1\}$  a vector field  $\bar{X}_1$  to fulfill the following conditions:

(a)  $\bar{F}_0 = (0, 0, 0)$  and  $\bar{F}_1 = (0, 0, 1)$  are hyperbolic sources and  $\bar{S} = (0, 0, \frac{1}{2})$  is a hyperbolic saddle.

(b)  $\bar{X}_1(re^{i\theta}, 0) = \bar{X}_1(re^{i\theta}, 1)$ .

(c) If  $\bar{X}_1(re^{i\beta_0}, s) = (v_1(re^{i\beta_0}), v_2(s))$  then  $\bar{X}_1(re^{-i\beta_0}, s) = (-v_1(re^{i\beta_0}), v_2(s))$ .

(d)  $\bar{X}_1 \mid (\{e^{-i\beta_0}\} \times [0, 1] \cup \{e^{i\beta_0}\} \times [0, 1]) = 0$  and  $\{e^{-i\beta_0}\} \times [0, 1] \cup \{e^{i\beta_0}\} \times [0, 1]$  is a normally hyperbolic set for  $\bar{X}_1$ . Now we define on  $D^2 \times S^1$  the vector field  $X_1(z, s) = \bar{X}_1(z, s \pmod{1})$ . Let  $\varepsilon > 0$  be small and  $\lambda: [0, 1] \rightarrow [0, 1]$  be the  $C^\infty$  function given by Figure 5.

The vector field  $E_1(re^{i\beta}, e^{i\gamma}) = ((1 - \lambda(r))\varepsilon ie^{i\beta}, \lambda(r)\varepsilon ie^{i\gamma})$  is equivariant under the fixed point action.  $Y = X_1 + E_1$  is a vector field defined on  $D^2 \times S^1$  with the following properties (Figure 6):

(1)  $Y$  is equivariant under the action with fixed points.

(2)  $S$  and  $F$  are hyperbolic singularities.

(3)  $\partial(D^2 \times S^1)$  is a normally hyperbolic set [HPS].

(4) The longitudes of  $\partial(D^2 \times S^1)$  are closed trajectories.

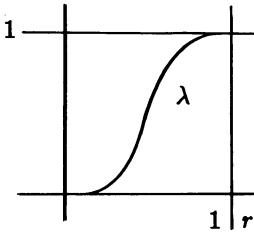


FIGURE 5

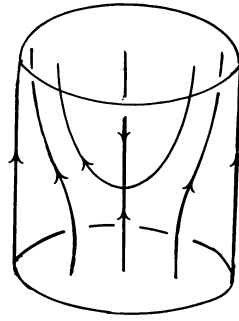


FIGURE 6

Our last construction will be a vector field equivariant under the principal action on  $D^2 \times S^1$ .  $X_0$  as defined earlier is also equivariant under the principal action (the condition  $X_0(z z_1) = X_0(z_1)$  is no longer needed). Again the vector field  $E_2(re^{i\beta}, e^{i\lambda}) = (\lambda(r)\varepsilon i e^{i\beta}, (1-\lambda(r))\varepsilon i e^{i\gamma})$ ,  $\lambda$  as before and  $\varepsilon$  small, is also invariant under the principal action. Finally define the vector field  $V = X_0 + E_2$  (Figure 7).

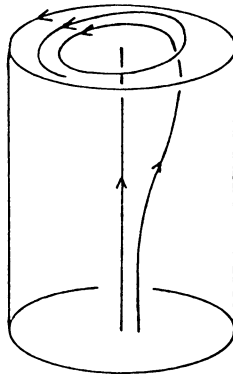


FIGURE 7

It has the following properties:

- (1) It is equivariant under the principal action.
- (2)  $\{0\} \times S^1$  is a hyperbolic closed orbit.
- (3)  $\partial(D^2 \times S^1)$  is a normally hyperbolic set.
- (4) The meridians of  $\partial(D^2 \times S^1)$  are closed orbits of  $V$  and then transverse to the  $G$  orbits.

(2) and (3) come from the persistence of normally hyperbolic sets [HPS].

(b) *Construction of the vector fields on  $S^2 \times S^1$  and  $S^3$ .* Let us decompose  $S^2 \times S^1 = T_0 \cup_i T_1$  where  $T_0 = T_1 = D^2 \times S^1$  and  $i: \partial T_0 \rightarrow \partial T_1$ ,  $i(x, y) = (x, y)$  is the identification along the boundary. If we take the action with fixed points on both  $T_0$  and  $T_1$  we have a  $G$  action defined on  $S^2 \times S^1$ . The set of fixed points of this action are  $\{s\} \times S^1$  and  $\{n\} \times S^1$  where  $s$  and  $n$  are the south and north poles of  $S^2$ . We define the vector field  $Z$  on  $S^2 \times S^1$  by  $Z|_{T_i} = X$  (defined in Example (a)),  $i = 1, 2$ . This vector field is equivariant under the  $G$  action just defined, it has  $\{s\} \times S^1$  and  $\{n\} \times S^1$  as expanding hyperbolic closed orbits and  $\partial T_0 = \partial T_1$  as

a normally hyperbolic attracting torus. All trajectories of  $Z \mid \partial T_0$  are closed and transverse to the  $G$  orbits.

As for the sphere we take  $S^3 = T_0 \cup_f T_1$  where  $f: \partial T_0 \rightarrow \partial T_1$ ,  $f(x, y) = (y, x)$ . If on  $T_0$  we take the principal action and on  $T_1$  the action with fixed points then, via the identification  $f$ , they glue together to give a  $G$  action on  $S^3$ . Now we define a vector field  $W$  on  $S^3$  by  $W \mid T_0 = V$  and  $W \mid T_1 = Y$  ( $V$  and  $Y$  defined in Example (a)).  $W$  has one expanding hyperbolic closed orbit in the interior of  $T_0$ , one saddle and one source both hyperbolic and inside  $T_1$  and  $\partial T_0 = \partial T_1$  is a normally hyperbolic attracting torus. On  $\partial T_0$  all trajectories of  $W$  are closed and transverse to the  $G$  orbits.

It follows from §4(II) that  $Z$  and  $W$  have modulus of stability one.

(c) *Construction of the vector field on any 3-manifold.* Let  $T$  be an equivariant tubular neighborhood of an orbit  $\eta$  in  $P$  (which is open and dense in any 3-manifold where  $G$  acts). Taking  $D^2 \times S^1$  with the principal action  $z(z_1, z_2) = (z_1, zz_2)$  there is an equivariant diffeomorphism  $\psi: T \rightarrow D^2 \times S^1$ ,  $\psi(\eta) = 0 \times S^1$ . The function  $\tilde{f}: D^2 \times S^1 \rightarrow R$  given by  $\tilde{f}(x, y, \theta) = x^2 + y^2$  is invariant under the principal action. Extend  $\tilde{f} \circ \psi: T \rightarrow R$  to get a  $C^\infty$   $G$  invariant function  $f: M \rightarrow R$ . Using Theorem D there is close to  $f$  a  $C^\infty$  function  $g: M \rightarrow R$  such that  $Y = \text{grad } g \in GMS$ . There is an orbit  $\eta_1$  close to  $\eta$  such that  $\psi_1: T_1 \rightarrow D^2 \times S^1$ ,  $\psi_1(\eta_1) = 0 \times S^1$  is equivariant and  $f \circ \psi_1^{-1}(x, y, \theta) = ax^2 + by^2$ ,  $a$  and  $b$  close to 1. On  $D^2 \times S^1$  we take a vector field  $\tilde{Z}$  such that

(a)  $\tilde{Z}$  is equivariant under the principal action.

(b) The torus  $T_2 = \{z: |z| = 1/2\} \times S^1$  is a normally hyperbolic source for  $\tilde{Z}$  and the trajectories of  $\tilde{Z} \mid T_2$  are transverse to the orbits of the action.

(c)  $0 \times S^1$  is a normally hyperbolic sink for  $\tilde{Z}$ .

Let  $\psi: M \rightarrow R$  be a  $C^\infty$  function which is zero on  $\psi_1(\{z \in D^2: |z| \leq \frac{3}{4}\} \times S^1)$  and it is 1 on  $M - \dot{T}$ . If  $Z$  is the pull back by  $\psi_1$  of  $\tilde{Z}$  define  $W = (1 - \psi)Z + \psi Y$ .  $W$  has, by Theorem C, nonzero finite moduli of stability.

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UNIVERSIDADE FEDERAL DE GOIÁS, DEPARTAMENTO DE MATEMÁTICA, 74000-GOIÂNIA-GOIÁS, BRASIL

PONTIFÍCIA UNIVERSIDADE CATÓLICA, DEPARTAMENTO DE MATEMÁTICA, 22450-RIO DE JANEIRO-RJ, BRASIL